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THE METHOD OF R-FUNCTIONS IN COMPUTING FIELDS FOR BODIES WHOSE PHYSICAL CHARACTERISTICS HAVE FIRST-ORDER DISCONTINUITIES*

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A new method is given, recognising the contact conditions appearing in bodies composed of elements with different physical characteristics. The method is based on representing the solution structures by various analytic expressions in the corresponding subregions. Two types of contact conditions appearing in the problems of computing electric fields for the locally isotropic and anisotropic conductivities are considered. The Ritz method is used to obtain a solution, and numerical results are given.

In considering the problems of thermal conductivity, electrostatics, and the theory of elasticity for bodies composed of elements with different physical characteristics, we find that additional conditions arise in addition to the usual boundary conditions at the boundary of the body, namely contact conditions within the region Ω (Fig. 1). The form of these conditions is determined by the physical formulation of the problem. Various types of contact conditions were considered in /1-5/; in all cases, however, after constructing the solution structure $u = B(\Phi)$, taking all boundary conditions at the outer boundary $\partial\Omega$ into account, the following transformation retaining the boundary $\partial\Omega$ was carried out:

$$Q = \{x^1 = x + \omega^2(x)\alpha(x)\} \tag{1}$$
$$x^1 = \{x_i^1\},\ x = \{x_i\},\ \alpha = \{\alpha_i\};\ i = 1, 2$$

where $\omega = 0$ is the equation of the outer boundary and α_i are functions chosen in a special manner depending on the form of the contact conditions. In practice, we must check every time whether the mapping of the transformation Q belongs to the region $\Omega \cap \partial\bar{\Omega}$. Additional difficulties are caused by the deformation of the spline mesh under the transformation Q, complicating the computations in quadratures.

Below we present a different method of including the contact conditions, based on representing the solution structures by different analytic expressions in different subregions.

Let us consider the equation

$$\operatorname{div}(\varepsilon_i \operatorname{grad} u) = 0 \tag{2}$$

in a finite region Ω (Fig. 2) with a piecewise homogeneous inclusions and mixed type boundary conditions

$$u|_{\partial\Omega_1} = 0,\quad u|_{\partial\Omega_2} = 1,\quad \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega_3} = 0 \tag{3}$$

and contact conditions

$$u_i|_{\partial\Omega_{ij}-0} = u_j|_{\partial\Omega_{ij}+0},\quad \varepsilon_i \left.\frac{\partial u_i}{\partial n_i}\right|_{\partial\Omega_{ij}-0} = \varepsilon_j \left.\frac{\partial u_j}{\partial n_i}\right|_{\partial\Omega_{ij}+0} \tag{4}$$

In this case we can represent the solution structure in the form

$$u_i = B(\Phi) + \delta_i\bar{\omega}_i D_1^{(i)} B(\Phi),\ \bar{\omega}_i = \omega_0^2 \wedge \omega_i \tag{5}$$

where $\omega_0 = 0$ is the equation of the outer boundary, $\omega_i = 0$ are normalized equations of the boundary $\partial\Omega_i$ of the subregions of Ω (we allow the exclusion of the segments belonging to $\partial\Omega$), δ_i are constants determined by the conditions (4), $B(\Phi)$ is the structure of the solution taking into account the boundary conditions at the outer boundary $\partial\Omega$ [5]. From (5) it follows that the first condition of (4) is satisfied automatically by virtue of the fact that $\omega_i = 0$ on $\partial\Omega_i$. To satisfy the second condition of (4) we continue $\partial u/\partial n_i$ into the corresponding subregions using the operators $D_1^{(i)}$ [5]. Substituting (5) into (4), we obtain

$$\varepsilon_i [D_1^{(i)} B(\Phi) + \delta_i D_1^{(i)} B(\Phi)] = \varepsilon_j [D_1^{(i)} B(\Phi) - \delta_j D_1^{(j)} B(\Phi)]\quad D_1^{(i)} B = -D_1^{(j)} B$$

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and hence

$$\varepsilon_i(1+\delta_i)=\varepsilon_j(1+\delta_j),\quad \frac{\varepsilon_i}{\varepsilon_j}=\frac{1+\delta_j}{1+\delta_i}.\quad \delta_i=M\varepsilon_i-1 \tag{6}$$

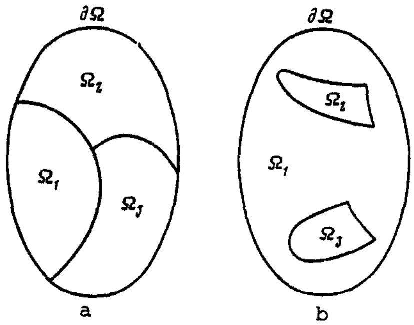

Fig.1

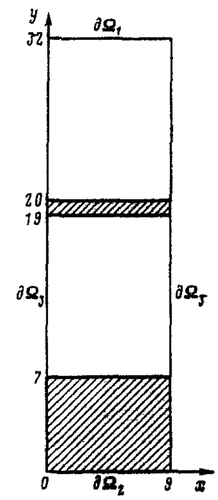

Fig.2

Since the ratio $\varepsilon_i/\varepsilon_j$ is the only important factor in satisfying the contact conditions, it follows that we can impose additional constraints on the quantities δ_k in the light of e.g. the results of numerical computations. In particular, we require that the following conditions be satisfied:

$$-1\leqslant M\varepsilon_j-1\leqslant 1,\ 0\leqslant M\leqslant 2/\varepsilon_i$$
$$-1\leqslant M\varepsilon_i-1\leqslant 1,\ 0\leqslant M\leqslant 2/\varepsilon_i$$

Thus the choice of δ_i, δ_j will depend on the constraint

$$0\leqslant M\leqslant \min\{2/\varepsilon_k\}\ (k=1,2,\ldots,K)$$

The structure (5) is valid for arbitrary distribution of the subregions with different physical characteristics, although it can be simplified in certain special cases. For example, in the case of the distribution of subregions shown in Fig.1,a,b we can write

$$u_1=B(\Phi),\ u_2=B(\Phi)+\delta_2\overline{\omega}_2D_1^{(2)}B(\Phi),\ u_3=B(\Phi)+\delta_3\overline{\omega}_3D_1^{(3)}B(\Phi)$$

Problem (2)-(4) was solved for $\varepsilon_1=1,\ \varepsilon_2=100$ and $\varepsilon_1=100,\ \varepsilon_2=1,$ i.e. for large differences in the electrical conductivity.

The undefined component Φ appearing in the structure of solution /5/ was written in the form $\Phi=c_1\chi_1+c_2\chi_2+\ldots+c_n\chi_n$ where $\chi_k=\chi_k(x,y)\ (k=1,2,\ldots,n)$ is a system of coordinate functions complete in $\Omega_1\cup\Omega_2$. The unknown coefficients c_K can be found using one of the variational or projection methods (the Ritz method is used here). To use the Ritz method, we pass to a boundary value problem with homogeneous boundary conditions and construct on the lineal of functions satisfying these conditions a functional, equivalent to the boundary value problem in question. Having proved the positive definiteness of the corresponding operator, we can apply the Ritz method, having also ensured the convergence in energy terms, to the exact solution. The passage to homogeneous boundary conditions is carried out by means of the substitution $u=u'+u_0$ where u_0 satisfies the boundary conditions (3).

Thus we seek a solution in the region $\Omega_1\cup\Omega_2$ of the following boundary value problem equivalent to (2)-(4):

$$Lu'=-Lu_0$$
$$\left.\frac{\partial u'}{\partial n}\right|_{\partial\Omega_H}=0,\quad u'|_{\partial\Omega_D}=0,\quad u_1'|_{\partial\Omega_{1,2}}=u_2'|_{\partial\Omega_{1,2}},\quad \varepsilon_1\left.\frac{\partial u_1'}{\partial n}\right|_{\partial\Omega_{1,2}}=\varepsilon_2\left.\frac{\partial u_2'}{\partial n}\right|_{\partial\Omega_{1,2}}$$

Let us show the positive definiteness of the operator L on the lineal U. We have

$$(Lu', u') = -\varepsilon_1 \int_{\Omega_1} \triangle u_1' u_1' \, d\Omega_1 - \varepsilon_2 \int_{\Omega_2} \triangle u_2' u_2' \, d\Omega =$$
$$\varepsilon_1 \int_{\Omega_1} (\nabla u_1')^2 \, d\Omega_1 + \varepsilon_2 \int_{\Omega_2} (\nabla u_2')^2 \, d\Omega_2 - \varepsilon_1 \int_{\partial\Omega_1 \cup \partial\Omega_{1,2}} u_1' \frac{\partial u_1'}{\partial n_1} \, d\partial\Omega -$$
$$\varepsilon_2 \int_{\partial\Omega_1 \cup \partial\Omega_{1,2}} \frac{\partial u_2'}{\partial n_2} \, d\partial\Omega$$
$$\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2, \ \partial\Omega_1 = \partial\Omega_{1D} \cup \partial\Omega_{1H}, \quad \partial\Omega_2 = \partial\Omega_{2D} \cup \partial\Omega_{2H}$$

Taking into account the homogeneous Dirichlet boundary conditions on the contour $\partial\Omega_{1D}$ and $\partial\Omega_{2D}$, Neuman conditions on the contour $\partial\Omega_{1H}, \partial\Omega_{2H}$ and contact conditions on $\partial\Omega_{1,2}$, we obtain

$$(Lu', u') = \varepsilon_1 \int_{\Omega_1} (\nabla u_1')^2 \, d\Omega_1 + \varepsilon_2 \int_{\Omega_2} (\nabla u_2')^2 \, d\Omega_2$$

for $(\varepsilon_1 \vee \varepsilon_2) > 0$, and by virtue of the Friedrichs inequality, [4] $(Lu', u') \geqslant \gamma^2 \| u' \|^2$.

We have used the cubic Schönberg splines as the approximation functions (the order of the approximation space $n = 670$ corresponds to line a, and 1310 to line b in Fig. 3). The maximum normed error over the region does not exceed 1% ($u_{T1,2}$ are exact solutions for $\varepsilon_1 = 1, \varepsilon_2 = 100$; $\varepsilon_1 = 100, \varepsilon_2 = 1$). When the Chebyshev polynomials ($n = 45$) and splines ($n = 49$) are used, the error amounts to 12%. Such poor accuracy is explained by the large difference in the sizes of the characteristic subregions. A sufficiently dense mesh of splines produces good results.

Problem (2)-(4) was also solved for the region shown in Fig. 4. Numerical computations were carried out using the same approximation techniques as in the previous problem. The results are shown by means of the level lines in Fig. 5 for $\varepsilon_1 = 1, \varepsilon_2 = 100$ (a) and $\varepsilon_1 = 100, \varepsilon_2 = 1$ (b).

All computations were carried out using the automatic programming system for the digital BESM-6 computer, developed at the Engineering Problems Institute of the Academy of Sciences of the UKrSSR. The passage from the problem corresponding to Fig. 2 to the problem corresponding to Fig. 4 required changing four punched cards containing the information on the geometry of the region boundaries and the separation of the media.

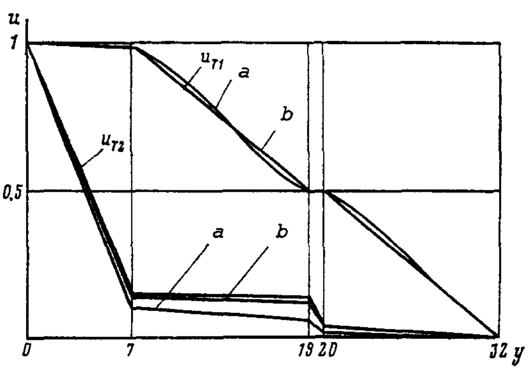

Fig. 3

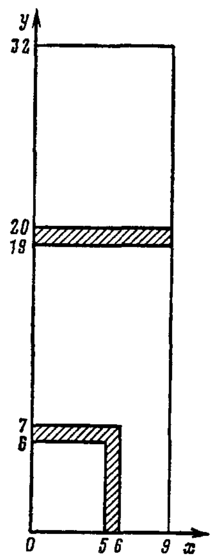

Fig. 4

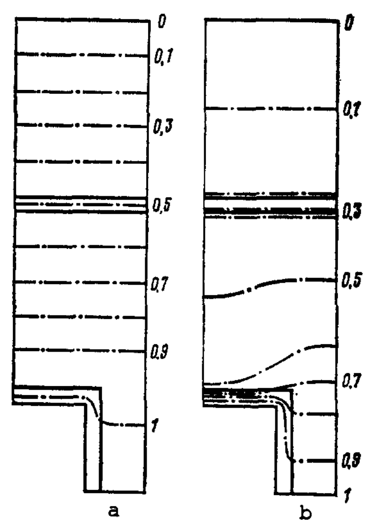

Fig. 5

The contact conditions for the problem studied in /4/ have the form

$$u_i |_{\partial\Omega_{ij}-0} = u_j |_{\partial\Omega_{ij}+0} \tag{7}$$
$$-\sigma_i \frac{\partial u_i}{\partial n_i} + \gamma_i \frac{\partial u_i}{\partial \tau_i} \Big|_{\partial\Omega_{ij}-0} = -\sigma_j \frac{\partial u_j}{\partial n_i} + \gamma_j \frac{\partial u_j}{\partial \tau_i} \Big|_{\partial\Omega_{ij}+0}$$

Let us write the new version of the structure

$$u_i = B(\Phi) + \bar{\omega}_i[\delta_i D_1^{(i)} B(\Phi) + \varkappa_i T_1^{(i)} B(\Phi)] \tag{8}$$

We have retained here the notation used in formula (5). We obtain the quantities $\varkappa_i$ and δ_i from condition (7) as follows:

$$-\sigma_i[D_1^{(i)}B(\Phi) + \delta_i D_1^{(i)}B(\Phi) + \varkappa_i T_1^{(i)} B(\Phi)] + \gamma_i T_1^{(i)} B(\Phi) =$$
$$-\sigma_i[D_1^{(i)}B(\Phi) + \delta_j D_1^{(i)}B(\Phi) + \varkappa_j T_1^{(i)} B(\Phi)] + \gamma_j T_1^{(i)} B(\Phi)$$
$$(D_1^{(i)}B = -D_1^{(j)}B,\ T_1^{(i)}B = -T_1^{(j)}B)$$

and hence

$$\frac{\sigma_i}{\sigma_j} = \frac{1+\delta_j}{1+\delta_i} \quad \left(0 \leqslant M \leqslant \min\left\{\frac{2}{\sigma_l}\right\},\ \ l = 1, 2, \ldots, L\right)$$
$$-\sigma_i\varkappa_i + \gamma_i = -\sigma_j\varkappa_j + \gamma_j, \quad \varkappa_i = \frac{-N+\gamma_i}{\sigma_i}, \quad \varkappa_j = \frac{-N+\gamma_j}{\sigma_j}$$
$$(-1 \leqslant \varkappa_l \leqslant 1,\ \ \min(\gamma_l - \sigma_l) \leqslant N \leqslant \min(\gamma_l + \sigma_l),\ \ l = 1, 2, \ldots, L)$$

Formula (8) enables us to apply the proposed method to the solution of the class of problems described in /4/.

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STATE OF STRESS OF A PLANE WITH A PERIODIC SYSTEM OF PARALLEL PAIRS OF LONGITUDINAL SHEAR CRACKS*

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A closed solution of the problem of a periodic system of parallel pairs of collinear longitudinal shear cracks is obtained by the method of triple integral equations. The case of one crack of finite length in a band of periods was examined in /1-5/ for different states of stress, and of two semi-infinite cracks in /6,7/. The problem of two collinear cracks in an infinite medium was investigated in /8-11/.

Let an unlimited elastic plane xOy be weakened by a periodic system of slits $a \leqslant |x| \leqslant b$, $y = (2n+1)d$, $n = 0, \pm1, \pm2, \ldots$ The relationships /12/

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0, \quad \sigma_{xz} = \mu\frac{\partial w}{\partial x}, \quad \sigma_{yz} = \mu\frac{\partial w}{\partial y} \tag{1}$$

should be satisfied outside the slits., where μ is the shear modulus, w is the displacement along the z axis, and σ_{xz} and σ_{yz} are stress tensor components. We assume that the displacement and stress are periodic functions of the y with period $2d$. Then the problem reduces to constructing the solution of (1) in the strip $-d < y < d$ that satisfies the boundary conditions

$$w(x, \pm d) = 0, \ |x| \leqslant a, \ |x| \geqslant b \tag{2}$$
$$\sigma_{yz}(x, \pm d) = -T(x), \quad a < |x| < b \tag{3}$$

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